# Correction to "Thin Operators in a von Neumann Algebra"

By CATHERINE L. OLSEN in Buffalo (N.Y., U.S.A.)

The proof of Proposition 2.2 in my paper, in *Acta Sci. Math.*, **35** (1973), 211—216 is incorrect. The following argument should be substituted for this proof. The author apologizes for this error.

Lemma. *Let $\mathscr{A}$ be a von Neumann algebra, and $\mathscr{I}$ a uniformly closed ideal in $\mathscr{A}$. Let $\varphi$ be an irreducible representation of $\mathscr{A}$ on a Hilbert space $\mathfrak{H}$, with $\varphi(\mathscr{I})\neq\{0\}$. Then there is a net of projections $\{E_\alpha\}\subset\mathscr{I}$ such that $\{\varphi(E_\alpha)\}$ converges strongly to the identity operator on $\mathfrak{H}$.*

Proof. The image $\varphi(\mathscr{I})$ is an irreducible algebra of operators on $\mathfrak{H}$ [4, p. 53, 2.11.3]. Hence $\varphi(\mathscr{I})$ is strongly dense in $\mathscr{B}(\mathfrak{H})$, so by the Kaplansky Density Theorem, the unit ball of $\varphi(\mathscr{I})$ is strongly dense in the unit ball of $\mathscr{B}(\mathfrak{H})$. Let $\{A_\alpha\}$ be a net of operators in $\mathscr{I}$ such that $\{\varphi(A_\alpha)\}$ is a net of non-zero operators in the unit ball of $\mathscr{B}(\mathfrak{H})$ converging weakly to $I_\mathfrak{H}$. We may assume $A_\alpha=A_\alpha^*$: taking adjoints is weakly continuous so $\varphi\{\frac{1}{2}(A_\alpha+A_\alpha^*)\}$ converges weakly to $I_\mathfrak{H}$.

Let $E_\alpha(\sigma)$ be the spectral measure for $A_\alpha$. Consider the set of projections

$$E_{\alpha n}=I_\mathfrak{H}-E_\alpha\left(-\frac{1}{n},\frac{1}{n}\right)\quad(n=1,2,\ldots).$$

These converge strongly with $n$ to the support projection of $A_\alpha$, and for each $\alpha n$, $E_{\alpha n}\in\mathscr{I}$ [2, p. 855 Lemma 4.1]. Since $\|E_{\alpha n}A_\alpha E_{\alpha n}-A_\alpha\|\leqq 1/n$, we have $\|\varphi(E_{\alpha n}A_\alpha E_{\alpha n})-\varphi(A_\alpha)\|\leqq 1/n$. The set $\{E_{\alpha n}\}$ is a net under the ordering: $\alpha n\succ\alpha' n'$ if $\alpha>\alpha'$ and $n>n'$. We claim that $\{\varphi(E_{\alpha n})\}$ converges strongly to $I_\mathfrak{H}$.

Choose an arbitrary $x\in\mathfrak{H}$ with $\|x\|=1$, and let $\varepsilon>0$ be given. There is a $\beta$ such that $|((I-\varphi(A_\alpha))x,x)|<\varepsilon$ for $\alpha>\beta$, and an $m$ such that $1/m<\varepsilon$. Note that $\varphi(A_\alpha)\leqq I$ implies $\varphi(E_{\alpha n}A_\alpha E_{\alpha n})\leqq\varphi(E_{\alpha n})$. Thus for $\alpha>\beta$, $n>m$,

$$((I-\varphi(E_{\alpha n}))x,x)\leqq((I-\varphi(E_{\alpha n}A_\alpha E_{\alpha n}))x,x)\leqq$$

$$\leqq|((I-\varphi(A_\alpha))x,x)|+|((\varphi(A_\alpha)-\varphi(E_{\alpha n}A_\alpha E_{\alpha n}))x,x)|\leqq 2\varepsilon.$$

Thus $\varphi(E_{\alpha n})$ converges strongly to $I_\mathfrak{H}$, and the Lemma is proved.

The proof of Proposition 2.2 is correct through the sentence "Set $y=\varphi(A)x-(\varphi(A)x, x)x$." After this the argument should read as follows:

The representation $\varphi$ restricted to $\mathscr{I}$ is irreducible on $\mathfrak{H}$. Thus the argument in [6, p. 61, Proposition 3.1] yields a projection $E\in\mathscr{P}$, $E\leqq I-P$, which satisfies the relations

$$I \leqq \|\varphi(E)x\|^2+\beta, \quad \varphi(E)y = 0,$$

where $\beta$ is a small positive number to be determined later. In other words, $\varphi(I-E)y=y$, $\|\varphi(I-E)x\|^2\leqq\beta$.

By the Lemma, there is a net of projections $\{E_\alpha\}\subset\mathscr{P}$ with $\varphi(E_\alpha)$ converging strongly to $I_{\mathfrak{H}}$. Then $F_\alpha=E_\alpha\vee P\in\mathscr{P}$, and since $\varphi(E_\alpha)\leqq\varphi(F_\alpha)$, the net $\{\varphi(F_\alpha)\}$ also converges strongly to $I_{\mathfrak{H}}$. Set $T_\alpha=(I-E)F_\alpha(I-E)$. Then $\{T_\alpha\}$ is a net of positive operators in $\mathscr{I}$ with

$$P \leqq T_\alpha \leqq rp(T_\alpha) \leqq I-E,$$

and $\{\varphi(T_\alpha)\}$ converges strongly to $\varphi(I-E)$. Observe that $rp(T_\alpha)\in\mathscr{P}$. For, if we set $S_\alpha=F_\alpha(I-E)$, then $T_\alpha=S_\alpha * S_\alpha$. Then $rp(S_\alpha)\leqq F_\alpha$, so $rp(S_\alpha)\in\mathscr{P}$. Thus the equivalent projection $rp(S_\alpha^*)$ is in $\mathscr{P}$. Finally, $rp(T_\alpha)\leqq rp(S_\alpha^*)$ implies $rp(T_\alpha)\in\mathscr{P}$. Since $\varphi(T_\alpha)\leqq$ $\leqq\varphi(rp(T_\alpha))\leqq\varphi(I-E)$, we have $\{\varphi(rp(T_\alpha))\}\subset\varphi(\mathscr{P})$ converges strongly to $\varphi(I-E)$. Hence we can find $Q\in\mathscr{P}$, $Q=rp(T_{\alpha_0})\geqq P$, such that

$$\|y-\varphi(Q)y\| = \|\varphi(I-E)y-\varphi(Q)y\| < \beta.$$

In particular $\|\varphi(Q)y\|>\|y\|-\beta$. Furthermore, since $\varphi(Q)\leqq\varphi(I-E)$, we have $\|\varphi(Q)x\|^2\leqq\|\varphi(I-E)x\|^2\leqq\beta$. Thus we have:

$$\begin{aligned}\|QA(I-Q)\| &\geqq \big\|\varphi(QA(I-Q))x\big\| \geqq \|\varphi(QA)x\| - \|\varphi(QA)\|\,\|\varphi(Q)x\| \geqq \\ &\geqq \big\|\varphi(Q)\big(\varphi(A)x-(\varphi(A)x, x)x\big)+\varphi(Q)\big(\varphi(A)x, x\big)x\big\| - \|A\|\sqrt{\beta} \geqq \\ &\geqq \|\varphi(Q)y\| - \big\|\varphi(Q)\big(\varphi(A)x, x\big)x\big\| - \|A\|\sqrt{\beta} \geqq \\ &\leqq \|y\| - \beta - \big|(\varphi(A)x, x)\big|\sqrt{\beta} - \|A\|\sqrt{\beta}.\end{aligned}$$

For sufficiently small choice of $\beta$ we obtain

$$\|QA(I-Q)\| \geqq \|\varphi(A)x-(\varphi(A)x, x)x\| -\varepsilon.$$

Hence the proof is complete.

*(Received February 1, 1974)*